

# Co-primeness preserving higher dimensional extension of $q$ -discrete Painlevé I, II equations

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## Abstract

We construct the  $q$ -discrete Painlevé I and II equations and their higher order analogues by virtue of periodic cluster algebras. Using particular  $k \times k$  exchange matrices, we show that the cluster algebras corresponding to  $k = 4$  and  $5$  give the  $q$ -discrete Painlevé I and II equations respectively. For  $k \geq 6$ , we have the higher order discrete equations that satisfy an integrable criterion, the co-primeness property.

## 1 Introduction

A cluster algebra introduced by Fomin and Zelevinsky is a commutative ring described by cluster variables and coefficients[1, 2]. It has a wide range of connections with various fields of mathematics and theoretical physics. One of the interesting connections is its relation to integrable systems. The cluster variables and coefficients obtained from a mutation of an initial seed satisfy certain difference equations, and some of them are related to discrete integrable equations. In particular, Hone-Inoue[3] and the author[4, 5] have shown that some periodic cluster algebras give the discrete Painlevé equations and its analogues of higher degrees. An important property of the cluster algebra is its Laurent phenomenon, that is, the cluster variables determined from initial seeds are always expressed as Laurent polynomials of the initial variables. These discrete Painlevé type equations of higher degrees also have Laurent phenomenon as is guaranteed by their construction. On the other hand, an important property of the discrete Painlevé equations is to have singularity confinement property[6]. Hence we naturally expect that these discrete Painlevé type equations also have singularity confinement property. However it is fairly difficult to check whether the discrete mappings of higher degrees have confined singularities or not due mainly to the increase of singularity patterns in the mappings with higher degrees. Recently, with the aim of refining integrability criteria, Kanki et al. have proposed the ‘irreducibility’ and the ‘co-primeness’ properties to distinguish integrable mappings[7]. Actually the co-primeness property is regarded as an algebraic reinterpretation of singularity confinement. Let us consider a discrete mapping with Laurent phenomenon. The mapping has the irreducibility, if every iterate is an irreducible Laurent polynomial of the initial variables. The equation satisfies the co-primeness condition, if every pair of two iterates is co-prime as Laurent polynomials.

In this article, we construct the Painlevé type equations from cluster algebras which include the  $q$ -Painlevé I and II equations, and show that they have the co-primeness property. In the next section, we briefly summarize necessary notion about cluster algebras. Generalized  $q$ -discrete Painlevé equations are constructed in section 3, and their co-primeness property is proved in section 4. Section 5 is devoted to the concluding remarks.

## 2 Seed mutations

In this section, we briefly explain the notion of cluster algebra[2] which we use in the following sections. Let  $\mathbf{x} = (x_1, x_2, \dots, x_N)$ ,  $\mathbf{y} = (y_1, y_2, \dots, y_N)$  be  $N$ -tuple variables. Each  $x_i$  is called a cluster variable and each  $y_i$  is called a coefficient. Let  $B = (b_{i,j})_{i,j=1}^N$  be a  $N \times N$  integer skew-symmetric matrix.  $B$  is called a exchange matrix. The triple  $(B, \mathbf{x}, \mathbf{y})$  is called a seed. A mutation is a particular transformation of seeds.

**Definition 2.1** [2] Let  $\mu_k : (B, \mathbf{x}, \mathbf{y}) \mapsto (B', \mathbf{x}', \mathbf{y}')$  ( $k = 1, 2, \dots, N$ ) be the mutation at  $k$ , defined as follows.

- New exchange matrix  $B' = (b'_{i,j})_{i,j=1}^N$  is defined from  $B$  as:

$$b'_{i,j} = \begin{cases} -b_{i,j} & (i = k \text{ or } j = k) \\ b_{i,j} + b_{i,k}b_{k,j} & (b_{i,k}, b_{k,j} > 0) \\ b_{i,j} - b_{i,k}b_{k,j} & (b_{i,k}, b_{k,j} < 0) \\ b_{i,j} & (\text{otherwise}) \end{cases}. \quad (2.1)$$

- New cluster variables  $\mathbf{x}' = (x'_1, x'_2, \dots, x'_N)$  are defined from  $B$  and  $\mathbf{x}$  as:

$$x'_i = \begin{cases} \frac{1}{x_k} \left( \prod_{b_{k,j} > 0} x_j^{b_{k,j}} + \prod_{b_{k,j} < 0} x_j^{-b_{k,j}} \right) & (i = k) \\ x_i & (i \neq k) \end{cases}. \quad (2.2)$$

- New coefficients  $\mathbf{y}' = (y'_1, y'_2, \dots, y'_N)$  are defined from  $B$  and  $\mathbf{y}$  as:

$$y'_i = \begin{cases} y_k^{-1} & (i = k) \\ y_i(y_k^{-1} + 1)^{-b_{k,i}} & (b_{k,i} > 0) \\ y_i(y_k + 1)^{b_{k,i}} & (b_{k,i} < 0) \\ y_i & (b_{k,i} = 0) \end{cases}. \quad (2.3)$$

For any seed  $t = (B, \mathbf{x}, \mathbf{y})$ , it holds that

$$\mu_k^2(t) = t. \quad (2.4)$$

For any seed  $t = (B, \mathbf{x}, \mathbf{y})$  and  $(i, j)$  such that  $b_{i,j} = 0$ , it holds that

$$\mu_i \mu_j(t) = \mu_j \mu_i(t). \quad (2.5)$$

The following theorem implies that the cluster algebras show the Laurent phenomenon.

**Theorem 2.2** [2] Let  $X(t)$  be the set of all the cluster variables obtained by iterative mutations to the seed  $t = (B, \mathbf{x}, \mathbf{y})$ . If  $x \in X(t)$ , then  $x \in \mathbb{Z}[\mathbf{x}^\pm]$ .

### 3 Generalized $q$ -discrete Painlevé I, II equations from mutations

For  $k \geq 4$ , we define  $k \times k$  exchange matrix  $B_k$  as

$$B_4 = \begin{bmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{bmatrix}, \quad (3.1)$$

$$B_5 = \begin{bmatrix} 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ -1 & 2 & 0 & -2 & 1 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -1 & -1 & 1 & 0 \end{bmatrix}, \quad (3.2)$$

$$B_k = \begin{bmatrix} 0 & -1 & 1 & & & 1 & -1 \\ 1 & 0 & -2 & 1 & & -1 & 1 \\ -1 & 2 & 0 & -2 & \ddots & & \\ & -1 & 2 & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & -2 & 1 \\ & & & \ddots & 2 & 0 & -2 & 1 \\ -1 & 1 & & & -1 & 2 & 0 & -1 \\ 1 & -1 & & & & -1 & 1 & 0 \end{bmatrix} \quad (k \geq 6). \quad (3.3)$$

These matrices  $B_k$  have the following form of mutation-period:

$$B_k \xrightarrow{\mu_1} RB_k R^{-1} \xrightarrow{\mu_2} R^2 B_k R^{-2} \xrightarrow{\mu_3} \dots \xrightarrow{\mu_k} R^k B_k R^{-k} = B_k, \quad (3.4)$$

where

$$R = \begin{bmatrix} 0 & & & & 1 \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}. \quad (3.5)$$

When we take the initial seed  $(B_k, \mathbf{x}, \mathbf{y})$ , where  $\mathbf{x} = (x_1, x_2, \dots, x_k)$ ,  $\mathbf{y} = (y_{0,1}, y_{0,2}, \dots, y_{0,k})$ , then new cluster variables  $x_n$  and coefficients  $y_{n,i}$  are defined as

$$\begin{aligned} & (x_1, x_2, \dots, x_k) \\ & \xrightarrow{\mu_1} (x_{k+1}, x_2, \dots, x_k) \\ & \xrightarrow{\mu_2} (x_{k+1}, x_{k+2}, \dots, x_k) \\ & \xrightarrow{\mu_3} \dots \\ & \xrightarrow{\mu_k} (x_{k+1}, x_{k+2}, \dots, x_{2k}) \\ & \xrightarrow{\mu_1} (x_{2k+1}, x_{k+2}, \dots, x_{2k}) \\ & \xrightarrow{\mu_2} (x_{2k+1}, x_{2k+2}, \dots, x_{2k}) \\ & \xrightarrow{\mu_3} \dots, \end{aligned} \quad (3.6)$$

$$\begin{aligned} & (y_{0,1}, y_{0,2}, \dots, y_{0,k}) \\ & \xrightarrow{\mu_1} (y_{1,1}, y_{1,2}, \dots, y_{1,k}) \\ & \xrightarrow{\mu_2} (y_{2,1}, y_{2,2}, \dots, y_{2,k}) \\ & \xrightarrow{\mu_3} \dots. \end{aligned} \quad (3.7)$$

The following proposition is readily obtained from the definition of mutation.

**Proposition 3.1** *For any  $n \in \mathbb{Z}$ , the cluster variables  $x_n$  satisfy the bilinear equation*

$$x_{n+k}x_n = x_{n+k-1}x_{n+1} + x_{n+k-2}x_{n+2}. \quad (3.8)$$

If  $k = 4$  and  $5$ , then the above bilinear equations are called the Somos-4 sequence and the Somos-5 sequence respectively[8]. From Theorem 2.2, we obtain the following proposition.

**Proposition 3.2** *For any  $n \in \mathbb{Z}$ , the cluster variables  $x_n$  are in  $\mathbb{Z}[x_1^\pm, x_2^\pm, \dots, x_k^\pm]$ .*

From the definition of mutation, the coefficients  $y_{n,i}$  satisfy the following relations:

$$\begin{cases} y_{n,n} = y_{n-1,n}^{-1} \\ y_{n,n+1} = y_{n-1,n+1}(y_{n-1,n} + 1) \\ y_{n,n+2} = y_{n-1,n+2}(y_{n-1,n}^{-1} + 1)^{-2} \\ y_{n,n+3} = y_{n-1,n+3}(y_{n-1,n} + 1) \end{cases} \quad (k = 4), \quad (3.9)$$

$$\begin{cases} y_{n,n} = y_{n-1,n}^{-1} \\ y_{n,n+1} = y_{n-1,n+1}(y_{n-1,n} + 1) \\ y_{n,n+2} = y_{n-1,n+2}(y_{n-1,n}^{-1} + 1)^{-1} \\ y_{n,n+3} = y_{n-1,n+3} \\ \vdots \\ y_{n,n+k-3} = y_{n-1,n+k-3} \\ y_{n,n+k-2} = y_{n-1,n+k-2}(y_{n-1,n}^{-1} + 1)^{-1} \\ y_{n,n+k-1} = y_{n-1,n+k-1}(y_{n-1,n} + 1) \end{cases} \quad (k \geq 5), \quad (3.10)$$

where we consider that an index  $i$  of the coefficients  $y_{n,i}$  takes a value in  $\mathbb{Z}/k\mathbb{Z}$ . By putting  $y_n := y_{n,n+1}$ , we obtain the following proposition.

**Proposition 3.3** *For any  $n \in \mathbb{Z}$ , the coefficients  $y_n$  satisfy*

$$y_{n+k}y_n = \frac{(y_{n+k-1} + 1)(y_{n+1} + 1)}{(y_{n+k-2}^{-1} + 1)(y_{n+2}^{-1} + 1)}. \quad (3.11)$$

### 3.1 Case of $k = 2m$ ( $m = 2, 3, \dots$ )

We consider the equation (3.11) for the case of  $k = 2m$  ( $m = 2, 3, \dots$ ). From deformation of the equation (3.11), we obtain

$$\frac{y_{n+k}y_{n+k-1}(y_{n+2} + 1)}{y_{n+2}y_{n+1}(y_{n+k-1} + 1)} = \frac{y_{n+k-1}y_{n+k-2}(y_{n+1} + 1)}{y_{n+1}y_n(y_{n+k-2} + 1)}. \quad (3.12)$$

Hence we find

$$\frac{y_{n+k-1}y_{n+k-2}(y_{n+1} + 1)}{y_{n+1}y_n(y_{n+k-2} + 1)} = \alpha, \quad (3.13)$$

where  $\alpha$  is a constant. Similarly we have

$$y_{n+k-1}y_{n+1} \prod_{i=1}^{k-3} \frac{y_{n+i+1}^2}{y_{n+i+1} + 1} = \alpha y_{n+k-2}y_n \prod_{i=1}^{k-3} \frac{y_{n+i}^2}{y_{n+i} + 1}, \quad (3.14)$$

and

$$y_{n+k-2}y_n \prod_{i=1}^{k-3} \frac{y_{n+i}^2}{y_{n+i} + 1} = \beta \alpha^n, \quad (3.15)$$

where  $\beta$  is another constant. Therefore, we obtain the following proposition.

**Proposition 3.4** *If  $k \geq 4$  is even, then the coefficients  $y_n$  satisfy the difference equation*

$$y_{n+k-2}y_n = \beta \alpha^n \prod_{i=1}^{k-3} \frac{y_{n+i} + 1}{y_{n+i}^2}. \quad (3.16)$$

In particular, for  $k = 4$ , this equation is the  $q$ -discrete Painlevé I equation[9]:

$$y_{n+2}y_n = \beta \alpha^n \frac{y_{n+1} + 1}{y_{n+1}^2}. \quad (3.17)$$

Difference equation (3.16) has a bilinear form.

**Proposition 3.5** *If  $k \geq 4$  is even and  $x_n$  satisfy the bilinear equation*

$$x_{n+k}x_n = z_n(x_{n+k-1}x_{n+1} + x_{n+k-2}x_{n+2}), \quad (3.18)$$

where

$$z_n = \beta^{1/(k-3)} \alpha^{(2n-k+2)/(2k-6)}, \quad (3.19)$$

then

$$y_n := \frac{x_{n+k-1}x_{n+1}}{x_{n+k-2}x_{n+2}} \quad (3.20)$$

satisfy the difference equation (3.16).

Note that the bilinear equation (3.18) is a non-autonomous generalization of the equation (3.8).

### 3.2 Case of $k = 2m + 1$ ( $m = 2, 3, \dots$ )

We consider the equation (3.11) for the case of  $k = 2m + 1$  ( $m = 2, 3, \dots$ ). From deformation of the equation (3.11), we obtain the relation (3.12) and have

$$\frac{y_{n+k-1}y_{n+k-2}(y_{n+1} + 1)}{y_{n+1}y_n(y_{n+k-2} + 1)} = \alpha^2, \quad (3.21)$$

where  $\alpha$  is a constant. Similarly we obtain

$$\frac{\prod_{i=0}^{2m-2} y_{n+i+2}}{\prod_{i=1}^{m-1} (y_{n+2i+1} + 1)} = \alpha^2 \frac{\prod_{i=0}^{2m-2} y_{n+i}}{\prod_{i=1}^{m-1} (y_{n+2i-1} + 1)}, \quad (3.22)$$

and

$$\frac{\prod_{i=0}^{2m-2} y_{n+i}}{\prod_{i=1}^{m-1} (y_{n+2i-1} + 1)} = \beta\gamma^{(-1)^n} \alpha^n, \quad (3.23)$$

where  $\beta, \gamma$  are constants. Therefore we find the following proposition.

**Proposition 3.6** *If  $k = 2m + 1 \geq 5$  is odd, then the coefficients  $y_n$  satisfy the difference equation*

$$y_{n+2m-2}y_n = \beta\gamma^{(-1)^n} \alpha^n \frac{\prod_{i=1}^{m-1} (y_{n+2i-1} + 1)}{\prod_{i=1}^{2m-3} y_{n+i}}. \quad (3.24)$$

By putting  $f_n := y_{2n}, g_n := y_{2n+1}$ , we obtain from (3.24),

$$\begin{cases} f_{n+m-1}f_n = \beta\gamma\alpha^{2n} \prod_{i=0}^{m-2} \frac{g_{n+i} + 1}{g_{n+i}} \prod_{i=1}^{m-2} \frac{1}{f_{n+i}} \\ g_{n+m-1}g_n = \beta\gamma^{-1}\alpha^{2n+1} \prod_{i=1}^{m-1} \frac{f_{n+i} + 1}{f_{n+i}} \prod_{i=1}^{m-2} \frac{1}{g_{n+i}} \end{cases}. \quad (3.25)$$

In particular, for  $k = 5$  ( $m = 2$ ), this equation is the  $q$ -discrete Painlevé II equation[10]:

$$\begin{cases} f_{n+1}f_n = \beta\gamma\alpha^{2n} \frac{g_n + 1}{g_n} \\ g_{n+1}g_n = \beta\gamma^{-1}\alpha^{2n+1} \frac{f_{n+1} + 1}{f_{n+1}} \end{cases}. \quad (3.26)$$

Difference equation (3.24) has a bilinear form.

**Proposition 3.7** *If  $k = 2m + 1 \geq 5$  is odd and  $x_n$  satisfy the bilinear equation (3.18) with*

$$\begin{aligned} z_{2n} &= (\beta/\gamma)^{1/(m-1)} \alpha^{(2n-m+1)/(m-1)}, \\ z_{2n+1} &= (\beta\gamma)^{1/(m-1)} \alpha^{(2n-m+2)/(m-1)}, \end{aligned} \quad (3.27)$$

then

$$y_n := \frac{x_{n+k-1}x_{n+1}}{x_{n+k-2}x_{n+2}} \quad (3.28)$$

satisfy the difference equation (3.24).

## 4 Co-primeness of generalized $q$ -discrete Painlevé I, II equations

We give some definitions necessary for the proof of co-primeness of the generalized  $q$ -discrete Painlevé equations.

**Definition 4.1** [7] *A Laurent polynomial  $f \in R := \mathbb{Z}[x_1^\pm, \dots, x_n^\pm]$  is irreducible if, for every decomposition  $f = gh$  ( $g, h \in R$ ), at least one of  $g$  or  $h$  is a unit in  $R$ .*

**Definition 4.2** [7] Two Laurent polynomials  $f_1, f_2$  are co-prime in  $R := \mathbb{Z}[x_1^\pm, \dots, x_n^\pm]$  if the following condition is satisfied: we have decompositions  $f_1 = g_1 h, f_2 = g_2 h$  in  $R$ , then  $h$  must be a unit in  $R$ .

**Definition 4.3** [7] Two rational functions  $f_1, f_2$  are co-prime in the field  $F := \mathbb{Q}(y_1, \dots, y_n)$  if the following condition is satisfied: let us express  $f_1, f_2$  as  $f_1 = g_1/g_2, f_2 = h_1/h_2$  where  $g_1, g_2, h_1, h_2 \in \mathbb{Q}[y_1^\pm, \dots, y_n^\pm]$ ,  $(g_1, g_2)$  and  $(h_1, h_2)$  are co-prime pairs of polynomials. Then, every pair of polynomials  $(g_i, h_j)$  ( $i, j = 1, 2$ ) is co-prime in the sense of Definition 4.2.

The following theorem is the main result of the present article:

**Theorem 4.4** For  $k = 4$ , we define rational functions  $y_n \in \mathbb{Q}(y_0, y_1, \alpha, \beta)$  by the difference equation (3.16). If  $|n - n'| > 2$ , then  $y_n$  and  $y_{n'}$  are co-prime.

For  $k = 5$ , we define rational functions  $y_n \in \mathbb{Q}(y_0, y_1, \alpha, \beta, \gamma)$  by the difference equation (3.24). If  $|n - n'| > 3$ , then  $y_n$  and  $y_{n'}$  are co-prime.

Accordingly we have two conjectures.

**Conjecture 4.1** For  $k \geq 6$  is even, we define rational functions  $y_n \in \mathbb{Q}(y_0, \dots, y_{k-3}, \alpha, \beta)$  by the difference equation (3.16). If  $|n - n'| > k - 2$ , then  $y_n$  and  $y_{n'}$  are co-prime.

**Conjecture 4.2** For  $k = 2m + 1 \geq 7$  is odd, we define rational functions  $y_n \in \mathbb{Q}(y_0, \dots, y_{2m-3}, \alpha, \beta, \gamma)$  by the difference equation (3.24). If  $|n - n'| > k - 2$ , then  $y_n$  and  $y_{n'}$  are co-prime.

We use the following lemma[7].

**Lemma 4.5** Let  $\{p_1, p_2, \dots, p_n\}$  and  $\{q_1, q_2, \dots, q_n\}$  be two sets of independent variables with the following properties:

$$p_i \in \mathbb{Z}[q_1^\pm, q_2^\pm, \dots, q_n^\pm], \quad (4.1)$$

$$q_i \in \mathbb{Z}[p_1^\pm, p_2^\pm, \dots, p_n^\pm], \quad (4.2)$$

$$q_i \text{ is irreducible as an element of } \mathbb{Z}[p_1^\pm, p_2^\pm, \dots, p_n^\pm], \quad (4.3)$$

for  $i = 1, 2, \dots, n$ . Let us take an irreducible Laurent polynomial

$$f(p_1, p_2, \dots, p_n) \in \mathbb{Z}[p_1^\pm, p_2^\pm, \dots, p_n^\pm], \quad (4.4)$$

and another Laurent polynomial

$$g(q_1, q_2, \dots, q_n) \in \mathbb{Z}[q_1^\pm, q_2^\pm, \dots, q_n^\pm], \quad (4.5)$$

which satisfies

$$f(p_1, p_2, \dots, p_n) = g(q_1, q_2, \dots, q_n). \quad (4.6)$$

In these settings, the function  $g$  is decomposed as

$$g(q_1, q_2, \dots, q_n) = p_1^{r_1} p_2^{r_2} \dots p_n^{r_n} \cdot \tilde{g}(q_1, q_2, \dots, q_n), \quad (4.7)$$

where  $r_1, r_2, \dots, r_n \in \mathbb{Z}$  and  $\tilde{g}(q_1, q_2, \dots, q_n)$  is irreducible in  $\mathbb{Z}[q_1^\pm, q_2^\pm, \dots, q_n^\pm]$ .

**Proposition 4.6** For  $k \geq 4$ , we define Laurent polynomials  $x_n \in \mathbb{Z}[x_1^\pm, x_2^\pm, \dots, x_k^\pm]$  by the bilinear equation (3.8). If  $n \neq n'$ , then  $x_n$  and  $x_{n'}$  are co-prime.

Note that all  $x_n$  are in  $\mathbb{Z}[x_1^\pm, x_2^\pm, \dots, x_k^\pm]$  from Proposition 3.2.

**Proof** Let  $L := \mathbb{Z}[x_1^\pm, \dots, x_k^\pm]$  and define the sequence  $(u_n)$  which satisfies (3.8) . by substituting  $x_1 = \dots = x_{k-1} = 1, x_k = t$ :

$$u_{n+k} u_n = u_{n+k-1} u_{n+1} + u_{n+k-2} u_{n+2}, \quad (4.8)$$

$$u_1 = \dots = u_{k-1} = 1, u_k = t. \quad (4.9)$$

We also define a sequence  $(v_n)$  (Fibonacci sequence):

$$v_n = v_{n-1} + v_{n-2}, \quad (4.10)$$

$$v_1 = v_2 = 1. \quad (4.11)$$

Clearly  $x_{k+1}$  is irreducible in  $L$ . From Lemma 4.5, we have

$$x_j = x_{k+1}^a f_{irr} \quad (j = k+2, \dots, 2k+1), \quad (4.12)$$

where  $a \in \mathbb{Z}_{\geq 0}$  and  $f_{irr}$  is irreducible in  $L$ . By substituting  $x_1 = \dots = x_{k-1} = 1, x_k = t$ , (4.12) turns to

$$u_j = u_{k+1}^a \tilde{f}_{irr} = (t+1)^a \tilde{f}_{irr} \quad (j = k+2, \dots, 2k+1). \quad (4.13)$$

Here  $\tilde{f}_{irr}$  is a Laurent polynomial of  $t$ . We can prove inductively

$$u_j = u_{j-1} + u_{j-2} = v_{j-k+1}t + v_{j-k} \quad (j = k+2, \dots, 2k-3). \quad (4.14)$$

Since  $v_i$  and  $v_{i+1}$  are mutually co-prime,  $u_j$  does not have a factor of  $t+1$ , that implies  $a = 0$  and  $x_j$  ( $j = k+2, \dots, 2k-3$ ) is irreducible.

A straightforward calculation gives

$$u_{2k-2} = u_{2k-3} + u_{2k-4}u_k = u_{2k-3} + u_{2k-4}t = v_{k-3}t^2 + (v_{k-2} + v_{k-4})t + v_{k-3}. \quad (4.15)$$

If we suppose that  $u_{2k-2}$  has the following form:

$$u_{2k-2} = (c_1t + c_2)(t+1), \quad (4.16)$$

then  $c_1 = v_{k-3}, c_1 + c_2 = v_{k-2} + v_{k-4}, c_2 = v_{k-3}$ , and  $v_{k-3} = 2v_{k-4}$  which is contradiction. Therefore  $u_{2k-2}$  is irreducible.

Irreducibility of  $x_{2k-1}, x_{2k}, x_{2k+1}$  are proved in a similar manner.

Nextly, from Lemma 4.5,

$$x_j = x_{k+1}^a f_{irr} = x_{k+2}^{b_{k+2}} \cdots x_{2k+1}^{b_{2k+1}} g_{irr} \quad (j = 2k+2, \dots, 3k+1), \quad (4.17)$$

where  $a, b_i \in \mathbb{Z}_{\geq 0}$ , and  $f_{irr}, g_{irr}$  are irreducible in  $L$ . If we suppose that  $x_j$  is not irreducible ( $j = 2k+2, \dots, 3k+1$ ), we have

$$x_j = x_{k+1}x_i h \quad (i = k+2, \dots, 2k+1). \quad (4.18)$$

Here  $h$  is a unit in  $L$ , that is, a monic Laurent monomial of  $(x_1, \dots, x_k)$ . Let  $(\tilde{x}_n)$  be the sequence obtained by (3.8) with initial condition  $x_1 = \dots = x_k = 1$ , then we have

$$\tilde{x}_j = 2\tilde{x}_i \quad (i = k+2, \dots, 2k+1). \quad (4.19)$$

However, we can prove inductively

$$\tilde{x}_n > 2\tilde{x}_{n-1} \quad (n \geq k+3), \quad (4.20)$$

which is contradiction. Therefore  $x_j$  ( $j = 2k+2, \dots, 3k+1$ ) are irreducible.

- (Proof of (4.20)):

Since  $\tilde{x}_{k+2} = 3, \tilde{x}_{k+3} = 7$ , it holds that  $\tilde{x}_{k+3} > 2\tilde{x}_{k+2}$ . For  $n \geq k+3$ , suppose that  $\tilde{x}_n > 2\tilde{x}_{n-1}$ . Then we have

$$\begin{aligned} \tilde{x}_{n+1} &= \frac{\tilde{x}_n \tilde{x}_{n-k+2} + \tilde{x}_{n-1} \tilde{x}_{n-k+3}}{\tilde{x}_{n-k+1}} \\ &> \frac{2(\tilde{x}_n \tilde{x}_{n-k+2} + \tilde{x}_{n-1} \tilde{x}_{n-k+3})}{\tilde{x}_{n-k+2}} \\ &> \frac{2(\tilde{x}_n \tilde{x}_{n-k+2} + 2\tilde{x}_{n-1} \tilde{x}_{n-k+2})}{\tilde{x}_{n-k+2}} \\ &= 2(\tilde{x}_n + 2\tilde{x}_{n-1}) \\ &> 2\tilde{x}_n. \end{aligned} \quad (4.21)$$

Hence (4.20) holds by induction.

For  $j \geq 3k + 2$ , we use Lemma 4.5 again and find

$$x_j = x_{k+1}^a f_{irr} = x_{k+2}^{b_{k+2}} \cdots x_{2k+1}^{b_{2k+1}} g_{irr} = x_{2k+2}^{c_{2k+2}} \cdots x_{3k+1}^{c_{3k+1}} h_{irr} \quad (j \geq 3k + 2), \quad (4.22)$$

where  $a, b_i, c_i \in \mathbb{Z}_{\geq 0}$  and  $f_{irr}, g_{irr}, h_{irr}$  are irreducible in  $L$ . The relation (4.22) implies  $a = 0, b_{k+2} = \cdots = b_{2k+1} = 0, c_{2k+2} = \cdots = c_{3k+1} = 0$ . Hence  $x_j$  is irreducible ( $j \geq 3k + 2$ ). Therefore  $x_n$  is irreducible in  $L$  for any  $n \in \mathbb{Z}_{\geq 0}$ . Furthermore, from (4.20),  $x_n/x_{n'}$  ( $n \neq n'$ ) is not a Laurent monomial,  $x_n$  and  $x_{n'}$  are mutually prime. ■

**Proposition 4.7** Let  $z_n$  be given by (3.19) or (3.27) with  $k \geq 4$ . For initial variables  $x_1, \dots, x_k$ , we define  $x_n \in \mathbb{Q}(\alpha, \beta, \gamma)(x_1, \dots, x_k)$  by (3.18). Then  $x_n$  and  $x_{n'}$  are mutually co-prime for  $n \neq n'$ .

**Proof** From (3.19) and (3.27),  $\alpha = \beta = \gamma = 1$  gives  $z_n = 1$ . Then Proposition 4.6 gives Proposition 4.7. ■

Finally we prove Theorem 4.1.

#### Proof of Theorem 4.4

From Proposition 4.7,  $x_j$  is an irreducible Laurent polynomial of  $(x_1, x_2, x_3, x_4)$ , and we easily find it is also irreducible in  $L(x_1^{\pm}, x_2^{\pm}, y_0^{\pm}, y_1^{\pm})$ . Hence for  $|n - n'| > 2$ ,  $y_n$  and  $y_{n'}$  are mutually co-prime in  $\mathbb{Q}(x_1, x_2, y_0, y_1, \alpha, \beta)$ . But  $y_n$  and  $y_{n'}$  depends only on  $(y_0, y_1, \alpha, \beta)$  and the conjecture is true for  $k = 4$ . Similarly we can prove in the case of  $k = 5$ . ■

For  $k \geq 6$ , a similar approach does not hold, however, we note that the conjectures 4.1 and 4.2 are true in  $\mathbb{Q}(x_1, x_2, \dots, x_k)$ .

## 5 Conclusion

In this article we introduced generalized  $q$ -Painlevé equations (3.16), (3.24) with a parameter  $k \in \mathbb{Z}_{\geq 4}$ . These  $q$ -discrete equations are extension of the  $q$ -discrete Painlevé equations in the sense that they coincide with  $q$ -Painlevé equation I and II for  $k = 4$  and  $k = 5$  respectively and that they have co-primeness property which is regarded as an algebraic reinterpretation of singularity confinement. We conjecture that the same property holds for  $k \geq 6$ , and wish to investigate the conjectures 4.1 and 4.2. Extension of the results in this article to other discrete Painlevé equations is one of the problems we wish to address in the future.

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